TILBURG UNIVERSITY
KATHOLIEKE UNIVERSITEIT BRABANT
POSTBOX 90153
5000 LE TILBURG
THE NETHERLANDS

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DEPARTMENT OF ECONOMICS

RESEARCH MEMORANDUM

INVARIANT SUBSPACES AND INVERTIBILITY PROPERTIES FOR SINGULAR SYSTEMS: THE GENERAL CASE

Ton Geerts

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Ton Geerts *),
Tilburg University, Dept. of Econometrics,
P.O. Box 90153, NL-5000 LE Tilburg, the Netherlands.

ABSTRACT

Open-loop definitions and properties of several subspaces for *general* singular systems are characterized by means of a fully algebraic distributional framework. Simple recursive algorithms for producing these spaces as well as related duality aspects turn out to follow directly from these definitions. Next, we provide definitions and conditions for **two** notions of left (right) invertibility of a *general* singular system in terms of our distributions, subspaces, and Rosenbrock's system matrix, and we show which conditions represent the 'gap' between our invertibility concepts. Finally, we prove that in many cases left (right) invertibility is equivalent to left (right) invertibility of the system matrix.

KEYWORDS

Singular system, impulsive-smooth distributions, strong controllability, duality, weak and strong left and right invertibility.

1. Introduction.

We consider linear time-invariant systems on $\mathbb{R}^+ := [0, \infty)$ in the generalized state space form

$$E\dot{x}(t) = Ax(t) + Bu(t), \tag{1.1a}$$

$$y(t) = Cx(t) + Du(t), \tag{1.1b}$$

where $E, A \in \mathbb{R}^{l \times n}$, $B \in \mathbb{R}^{l \times m}$, $C \in \mathbb{R}^{r \times n}$, $D \in \mathbb{R}^{r \times m}$, and $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^r$ for all $t \geq 0$. No assumptions will be made on E or on the matrix pencil (sE - A). Systems (1.1) are called singular [1] - [4], implicit [5] - [6], descriptor systems [7] - [8], degenerate [9] or generalized systems [10]. Various contributors on singular systems have investigated various aspects under various assumptions - for the sake of brevity, we refer to our own references as well as to those mentioned there.

In this paper we will define and characterize several subspaces of $\mathbb{R}^n$ for *general* singular systems (1.1). Since the open-loop definitions of these spaces are in terms of (special) distributions, their systemic interest (e.g. in view of optimal control problems) becomes directly apparent. Our distributional framework enables us to formulate and prove in a straightforward manner various statements on these spaces and our algorithms for computing them are in line with earlier expectations (e.g. [10]). Moreover, we will present definitions of and equivalent statements (expressed in subspaces and Rosenbrock's system matrix [24]) on our concepts of *weak* and *strong* left and right invertibility for a system (1.1), and we will specify when the two notions are equivalent as in [23]. To the best of our knowledge, our results on invertibility for continuous-time singular systems are the most general and, perhaps, also the most elegant ones.

Before going into details in Section 2, we shall spend the rest of this Introduction on the issue of consistency of *initial conditions* and the interpretation of "initial conditions" in our distributional setting.

It is well known that every initial condition $x_0 := x(0)$ is *consistent* [1] if $l = n$ and **E** is invertible. In case of a singular matrix **E**, however, this need not be the case.

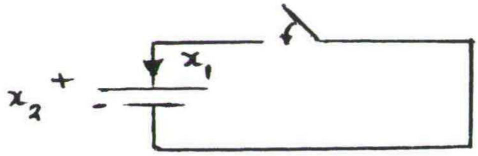
Example [3, p. 812].

Consider

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u.$$

It follows that $x_2 = -u$, $x_1 = -\dot{u}$. Hence, if u, sufficiently smooth, is given, then there exists only one consistent initial condition, namely $x_{01} = -\dot{u}(0^+)$, $x_{02} = -u(0^+)$. (Conversely, one can say that if x_{01}, x_{02} are given, then u is consistent if it is sufficiently well-behaved and $u(0^+) = -x_{02}$, $\dot{u}(0^+) = -x_{01}$).

However, when modeling e.g. electrical circuits, it may occur that the initial value x_0 need not be consistent, i.e., that $x_0 \neq x(0^+)$. For instance, in [3] it is stated that the model of our **Example** with u = 0 corresponds to a simple circuit with unit capacitor only, x_2 denoting its potential, x_1 the current (see below).



If at $t = 0$ the switch is closed and $x_{01} = x_1(0^-) = 0$, $x_{02} = x_2(0^-) \neq 0$ (and hence inconsistent), then the solution is [3] $x_2 = 0$, $x_1 = - x_{02}\delta(t)$, $\delta(t)$ denoting the Dirac delta function. In other words, for arbitrary initial conditions $x_0 := x(0^-)$ a solution of (1.1a) (if any) may exhibit *impulsive* behaviour even if the input u is an ordinary function.

Such observations led several authors on singular systems (e.g. [8]) to the use of generalized functions (*distributions* [11]), whereas others (e.g. [3]) based their analysis on the Laplace transformation approach of Doetsch [12, § 22].

Recently [13], [30] it was demonstrated that both viewpoints can be captured in one fully algebraic and therefore easily understandable distributional framework without using Kronecker canonical forms, state space decompositions, unnecessarily involved distributions or artificial extra parameters. The method's power lies in the combination of the linear system structure and the elegant class C_{imp} of allowed distributions. Loosely speaking (for more details, see Section 2), an element of C_{imp} is a linear combination of an *impulse* (a distribution with support 0) and a distribution that can be identified with a *smooth* function on $\mathbb{R}^+$ [15], and C_{imp} is a commutative algebra over $\mathbb{R}$ with convolution of distributions as multiplication (unit element δ, the Dirac delta distribution), see [14]. Instead of (1.1a), we introduce in [13], [30] its **distributional** version

$$\delta^{(1)}*Ex = Ax + Bu + Ex_0\delta, \qquad (1.2)$$

with $x_0 \in \mathbb{R}^n$, $\delta^{(1)}$ denoting the distributional derivative of δ, and $*$ standing for convolution of distributions. If $u \in C^m_{imp}$ (the m-vector version of C_{imp}), then we can define for every pair (x_0, u) the solution set

$$S(x_0, u) := \{x \in C^n_{imp} \mid (\delta^{(1)}E - A\delta)*x = Bu + Ex_0\delta\} \qquad (1.3)$$

and x is called a solution of (1.2) associated with (x_0, u) if $x \in S(x_0, u)$. For many properties of our distributional setup, see Section 2. Here, we would like to highlight the presence of a **point** x_0 in the distributional **differential equation** (1.2).

If $l = n$ and E is invertible, then we may assume without loss of generality that $E = I$ and (1.2) reduces to

$$\delta^{(1)} * x = Ax + Bu + x_0\delta. \tag{1.4}$$

This distributional version of the ordinary differential equation $\dot{x} = Ax + Bu$ on $\mathbb{R}^+$ has been extensively studied in [15]; since $(\delta^{(1)}I - A\delta)$ is within $C_{imp}^{n\times n}$ invertible w.r.t. convolution with inverse corresponding to the smooth function $\exp(At)$ on $\mathbb{R}^+$, see [15, p. 375], one can easily see that for every x_0 and every smooth u the distributional differential equation (1.4) has exactly one smooth solution x, corresponding to the function

$$\exp(At)x_0 + \int_0^t \exp(A(t-\tau))Bu(\tau)d\tau \tag{1.5}$$

on $\mathbb{R}^+$. It follows that $x(0^+) = x_0$ - apparently, the arbitrary point x_0 plays the role of initial condition if u is smooth and $E = I$. In general, however, x_0 as well as $u \in C_{imp}^{m}$ may be arbitrary in (1.2); consequently, the value of x immediately after the impulse, $x(0^+)$, may be unequal to x_0. What is more, we will establish that not so much the property $x_0 = x(0^+)$ as its generalization $Ex_0 = Ex(0^+)$ is strongly related to the question of smoothness for solutions x of (1.2).

Our approach of defining subspaces in Section 3 clearly parallels the method followed in [15] - the claims in [15, Section 3] turn out to be special cases of ours. One of the main differences between singular systems and standard systems (systems with $E = I$), however, is the fact, that, unlike any solution x of (1.4), a solution x of (1.2) might be "more impulsive" than the control u is. Our answer to this extra difficulty is the Main Lemma, see Section 2. Part of our work generalizes results in [10] as well as statements in [16] - in particular, we do not preassume the existence of the transfer function. Section 4 contains our main contributions on invertibility for singular systems.

2. Preliminaries.

As was stated in the Introduction, the distributional framework based on C_{imp} allows a fully algebraic treatment of general singular systems - one might even forget about being involved with distributions at all. We will recall the headlines only; for more details, see [14], [11].

Let $\mathcal{D}_-$ be the space of test functions with upper-bounded support and let $\mathcal{D}_+'$ denote the dual space of real-valued continuous linear functionals on $\mathcal{D}_-$. Then the space $\mathcal{D}_+$ of test functions with lower-bounded support can be considered as a subspace of $\mathcal{D}_+'$ by the identification $\langle \psi, \varphi \rangle = \int_{-\infty}^{+\infty} \psi(t)\varphi(t)dt$, where $\langle u, \varphi \rangle$ stands for the value of $u \in \mathcal{D}_+'$ at $\varphi \in \mathcal{D}_-$. It can be shown that every $u \in \mathcal{D}_+'$ has lower-bounded support. The distributional derivative $u^{(1)}$ of $u \in \mathcal{D}_+'$ is defined $\langle u^{(1)}, \varphi \rangle := - \langle u, \dot{\varphi} \rangle$, $\dot{\varphi}$ denoting the ordinary derivative of $\varphi \in \mathcal{D}_-$. With "pointwise" addition and scalar multiplication and with the convolution $*$ as **multiplication,** $\mathcal{D}_+'$ is a commutative algebra [17, vol. 2] over $\mathbb{R}$ with unit element δ, defined by $\langle \delta, \varphi \rangle = \varphi(0)$ $(\varphi \in \mathcal{D}_-)$. Also, we have $u^{(1)} = u^{(1)} * \delta = (u * \delta)^{(1)} = u * \delta^{(1)}$. Any linear combination of δ and its derivatives $\delta^{(l)}$, $l \geq 1$, is called *impulsive*. A distribution $u \in \mathcal{D}_+'$ that can be identified with an ordinary function (u!) is called *smooth* on $\mathbb{R}^+$ if u is smooth on $\mathbb{R}^+$ [15] and zero elsewhere.

Linear combinations of impulsive distributions and smooth distributions on $\mathbb{R}^+$ will be called *impulsive-smooth* [15, Def. 3.1] and the set C_{imp} of these impulsive-smooth distributions is a subalgebra. In particular, this implies that C_{imp} is closed under differentiation (= convolution with $\delta^{(1)}$) and under integration (= convolution with the inverse of $\delta^{(1)}$, the Heaviside distribution H). Also the next property of C_{imp} is important.

Proposition 2.1 [14, Theorem 3.11].

Let $u \in C_{imp}$. Then there exists a $v \in C_{imp}$ such that $u*v = v*u = \delta$ (i.e., $v = u^{-1}$) if and only if $u \notin \mathcal{D}_+$.

Thus, every impulsive distribution $u \neq 0$ is invertible within $\mathcal{C}_{imp}$. Now if we define [14, Def. 3.1]

$$p := \delta^{(1)},\ p^k := p^{k-1}*p\ (k \geq 2),\ p^0 := \delta, \tag{2.1a}$$

$$p^{-1} := H,\ p^{-l} = p^{-(l-1)}*p^{-1}\ (l \geq 2), \tag{2.1b}$$

then it is easily seen that $p^{k+l} = p^k*p^l$ $(k,l \in \mathbb{Z})$ [14, Prop. 3.2] and thus $(p^k)^{-1} = p^{-k}$ and $(p^0)^{-1} = p^0 = \delta$; we will write $p^0 = 1$ and $\alpha\delta = \alpha$ $(\alpha \in \mathbb{R})$. From now on, convolution will be denoted by juxtaposition (recall that $\mathcal{C}_{imp}$ is a commutative algebra). Observe that the decomposition of $u \in \mathcal{C}_{imp}$ in an impulsive and a smooth part is **unique**. If $\mathcal{C}_{p-imp}$ denotes the subalgebra of pure impulses and $\mathcal{C}_{sm}$ the subalgebra of smooth distributions on $\mathbb{R}^+$ and if $u = u_1 + u_2$, $u_1 \in \mathcal{C}_{p-imp}$, $u_2 \in \mathcal{C}_{sm}$, then $u(0^+) := \lim_{t \downarrow 0} u_2(0^+)$. If $u \in \mathcal{C}_{imp}$ is smooth, and $\dot{u}$ stands for the distribution that can be identified with the ordinary derivative of u on $\mathbb{R}^+$, then one can easily show that

$$pu = \dot{u} + u(0^+) \tag{2.2}$$

(with $u(0^+) = u(0^+)\delta$!). In particular, $p0 = 0$ (the derivative of 0 is 0), but also $p^{-1}0 = p^{-1}(p0) = (p^{-1}p)0 = 0$, i.e., the primitive of 0 equals 0. Thus, $pu = 0 \Leftrightarrow u = 0 \Leftrightarrow p^{-1}u = 0$. More generally, we even have

Proposition 2.2.

If $u, v \in \mathcal{C}_{imp}$ and $uv = 0$, then either u and/or v equals 0.

Proof. If $u \notin \mathcal{D}_+$, then $v = 0$ and if $v \notin \mathcal{D}_+$, then $u = 0$ by Proposition 2.1. If u and v are both smooth, then the claim follows from Titchmarsh's Theorem [18, Th. 152].

Next, let $\mathcal{C}_f$ denote the set of *fractional impulses*:

$$\mathcal{C}_f := \{u \in \mathcal{C}_{imp} \mid u = u_1u_2^{-1},\ u_{1,2} \in \mathcal{C}_{p-imp},\ u_2 \neq 0\}. \tag{2.3}$$

If $u = u_1u_2^{-1}$, $v = v_1v_2^{-1}$ $(u_2, v_2 \neq 0)$ are both in $\mathcal{C}_f$, then $u + v = (u_1v_2 + u_2v_1)(v_2u_2)^{-1} \in \mathcal{C}_f$, $uv = u_1v_1(u_2v_2)^{-1} \in \mathcal{C}_f$ and $\mathcal{C}_f$ is again a subalgebra of $\mathcal{C}_{imp}$. Moreover, Proposition 2.3.

The commutative field C_f is isomorphic to the commutative field of rational functions $\mathbb{R}(s)$.

Proof. Let $\mathbb{R}[s]$ denote the integral domain (with unit element) of polynomials with real coefficients. Then it is clear that $\mathbb{R}[s]$ and C_{p-imp} are isomorphic (see (2.1)). Now $\mathbb{R}(s)$ and C_f can be identified with the quotient fields of $\mathbb{R}(s)$ and C_{p-imp}, respectively [17, vol. 1, § 13].

Corollary 2.4.

Let k_1, k_2 be any nonnegative integers and let $M^{k_1 \times k_2}(s)$, $M_f^{k_1 \times k_2}(p)$ denote the sets of $k_1 \times k_2$ matrices with elements in $\mathbb{R}(s)$ and C_f, respectively. If $T(s) \in M^{k_1 \times k_2}(s)$ and $T(p)$ is the corresponding distributional matrix in $M_f^{k_1 \times k_2}(p)$, then

$$\exists_{L(s) \in M^{k_2 \times k_1}(s)} : L(s)T(s) = I^{k_2} \Leftrightarrow \exists_{L \in C_{imp}^{k_2 \times k_1}} : LT(p) = I^{k_2},$$

and also

$$\exists_{R(s) \in M^{k_2 \times k_1}(s)} : T(s)R(s) = I^{k_1} \Leftrightarrow \exists_{R \in C_{imp}^{k_2 \times k_1}} : T(p)R = I^{k_1}.$$

In particular, $T(s)$ is left (right) invertible as a rational matrix if and only $T(p)$ is left (right) invertible as a matrix with elements in C_f.

Proof. Assume that $L(s)T(s) = I^1$, let $L(p)$ be the corresponding matrix with elements in C_f. Then $L(p)T(p) = I^1$ ($= I^1\delta$!) because of Proposition 2.3. Conversely, assume that $T(s)\xi(s) = 0$ for some l-vector of rational functions. It follows that $T(p)\xi(p) = 0$. Since C_{imp} is a commutative ring (even an integral domain with unit element δ, see Proposition 2.2), we establish that $\xi(p) = I^1\xi(p) = [LT(p)]\xi(p) = L[T(p)\xi(p)] = 0$, i.e., $\xi(s) = 0$ and hence $T(s)$ is left invertible as a rational matrix (for references on linear algebra and matrix computations, we refer to [17], [19], [20]). The proof for the second claim runs analogously.

We are ready for the **distributional** version of (1.1) on $\mathbb{R}^+$ (see (1.2))

$$pEx = Ax + Bu + Ex_0, \tag{2.4a}$$

$$y = Cx + Du, \tag{2.4b}$$

together with the solution set $S(x_0, u)$ for every pair $(x_0, u) \in \mathbb{R}^n \times C^m_{imp}$ ((1.3)). We stress that this way of *defining* a general singular system on $\mathbb{R}^+$ unifies e.g. [3] - [4], [8], [10], [12], [25], [28], but also the well-known [15] for standard systems (see Section 1). In addition, if the arbitrary point x_0 is consistent (see Section 1), then it can be proven [13, Th. 2.13], [30, Sec. 2] that (2.4a) has a functional solution x with $x(0^+) = x_0$. For instance, consider the distributional version of

Example (continued).
Consider $p\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix}u + \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\begin{bmatrix}x_{01}\\ x_{02}\end{bmatrix}$ and let u, smooth, be given. If $x_{01} = -\dot{u}(0^+)$ and $x_{02} = -u(0^+)$, then $x_2 = -u$ and $x_2(0^+) = x_{02}$ and $x_1 = p(-u) - x_{02} = -\dot{u}$ ((2.2)) and $x_1(0^+) = x_{01}$.
Note that $u = 0$ yields $x_2 = 0$, $x_1 = -x_{02}$, which agrees with [3] (see Section 1).

Apparently, *singular* systems, unlike standard systems, may generate impulsive solutions even if the inputs are smooth. We will deal with this aspect by means of the next *basic* result.

Main Lemma 2.5.

Let $x_0 \in \mathbb{R}^n$, $u = u_1 + u_2$, $u_1 \in C^m_{p-imp}$, $u_2 \in C^m_{sm}$, $x = x_1 + x_2 \in S(x_0, u)$, $x_1 \in C^n_{p-imp}$, $x_2 \in C^n_{sm}$. Then

$$pEx_1 + E(x_2(0^+)) = Ax_1 + Bu_1 + Ex_0, \tag{2.5a}$$

$$pEx_2 = Ax_2 + Bu_2 + E(x_2(0^+)). \tag{2.5b}$$

Proof. We have $pEx_1 + E(x_2(0^+)) + [E(px_2 - x_2(0^+))] = Ax_1 + Bu_1 + Ex_0 + [Ax_2 + Bu_2]$ and $px_2 - x_2(0^+)$ is smooth ((2.2)).

Corollary 2.6.

Let $u \in C^m_{sm}$, $x_0 \in \mathbb{R}^n$. If $x \in S(x_0, u) \cap C^n_{sm}$, then $Ex_0 = E(x(0^+))$.

Remark 2.7.

In [13, Prop. 3.5] it is proven that the converse of Corollary 2.6 is true if $(sE - A)$ is invertible as a rational matrix. In general, however, x may be impulsive even if $Ex_0 = E(x(0^+))$.
Example: $p\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}0 & 0\\1 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} + \begin{bmatrix}0\\0\end{bmatrix}u + \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}\begin{bmatrix}x_{01}\\x_{02}\end{bmatrix}$.
If $x_{01} = 0$, then $x_{01} = x_1(0^+)$ $(x_1 = 0)$, but x_2 may be arbitrary.

Remark 2.8.

In principle it is possible to allow distributional inputs that are linear combinations of impulses and distributions associated with more general functions with support on $\mathbb{R}^+$. However, the class of these distributions does not have such nice properties as C_{imp}, and, moreover, it is long recognized that smoothness requirements do not limit the possibilities for the treatment of feedback (pole placement, e.g. [4]), associated optimal control problems [15], [9], [8], [28], [21], geometric approaches and invertibility properties [15], [22], [10], [23], realization theory [5], [16], or solvability aspects [13], [30].

Remark 2.9.

By application of Kronecker's canonical form, it can be shown (e.g. [8]) that $(pE - A)$ is invertible within $C^{n\times n}_{imp}$ if and only if $\det(sE - A) \neq 0$. Note that this result follows *directly* from Proposition 2.3 (or Corollary 2.4). The combination of this result with Lemma 2.5 turns out to be a successful one in the sequel.

3. Weak unobservability and strong controllability.

Given the system Σ: $pEx = Ax + Bu + Ex_0$, $y = Cx + Du$, with $x_0 \in \mathbb{R}^n$ and $u \in C^m_{imp}$. The following definitions generalize associated concepts in [15, Section 3].

Definition 3.1.

A point x_0 is called **weakly unobservable** if there exists an input $u \in C^m_{sm}$ and a state trajectory $x \in S(x_0, u) \cap C^n_{sm}$ such that $y = 0$. The space of these points is denoted by $\mathcal{V}(\Sigma)$.
A point x_0 is called **strongly controllable** if there exists an input $u \in C^m_{p\text{-}imp}$ and a state trajectory $x \in S(x_0, u) \cap C^n_{p\text{-}imp}$ such that $y = 0$. The space of these points is denoted by $\mathcal{W}(\Sigma)$.
A point x_0 is called **distributionally weakly unobservable** if there exists an input $u \in C^m_{imp}$ and a state trajectory $x \in S(x_0, u)$ such that $y = 0$. The space of these points is denoted by $\mathcal{V}_d(\Sigma)$.
A point x_0 is called **weakly unobservable strongly controllable** if there exists an input $u \in C^m_{imp}$ and a state trajectory $x \in S(0, u)$ such that $y = 0$ and $Ex_0 = E(x(0^+))$. The space of these points is denoted by $\mathcal{R}(\Sigma)$.

For further use, we recall Rosenbrock's system matrix [24]

$$P_\Sigma(s) = \begin{bmatrix} A - sE & B \\ C & D \end{bmatrix}; \qquad (3.1)$$

$P_\Sigma(p)$ denotes the corresponding distributional matrix. The first theorem on the four subspaces of Definition 3.1 follows directly from the Main Lemma 2.5.

Theorem 3.2.

$$\mathcal{V}_d(\Sigma) = \mathcal{V}(\Sigma) + \mathcal{W}(\Sigma), \quad \mathcal{R}(\Sigma) = \mathcal{V}(\Sigma) \cap \mathcal{W}(\Sigma).$$

Proof. First statement. $\Leftarrow$ Trivial, by definition. $\Rightarrow$ Let x_0 be such that for certain $\begin{bmatrix} x \\ u \end{bmatrix} \in C^{n+m}_{imp}$, $P_\Sigma(p)\begin{bmatrix} x \\ u \end{bmatrix} = \begin{bmatrix} -Ex_0 \\ 0 \end{bmatrix}$. Write $x = x_1 + x_2$, $u = u_1 + u_2$, u_1 and x_1 impulsive, u_2 and x_2 smooth. It follows that $pEx_1 = Ax_1 + Bu_1 + E(x_0 - x_2(0^+))$, $Cx_1 + Du_1 = 0$ and hence $(x_0 - x_2(0^+)) \in \mathcal{W}(\Sigma)$. In addition, $pEx_2 = Ax_2 + Bu_2 + E(x_2(0^+))$, $Cx_2 + Du_2 = 0$ and hence $x_2(0^+) \in \mathcal{V}(\Sigma)$. We establish that $x_0 \in \mathcal{V}(\Sigma) + \mathcal{W}(\Sigma)$. Second statement. $\Leftarrow$ Let x_0 be such that $P_\Sigma(p)\begin{bmatrix} x_1 \\ u_1 \end{bmatrix} = \begin{bmatrix} -Ex_0 \\ 0 \end{bmatrix}$, x_1 and u_1 impulsive, and $P_\Sigma(p)\begin{bmatrix} x_2 \\ u_2 \end{bmatrix} = \begin{bmatrix} -Ex_0 \\ 0 \end{bmatrix}$, x_2 and u_2 smooth. Then $P_\Sigma(p)\begin{bmatrix} -x_1 + x_2 \\ -u_1 + u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, in other words, $x := -x_1 + x_2 \in S(0, u)$ with $u := -u_1 + u_2$, $Cx + Du = 0$ and $E(x(0^+)) = E(x_2(0^+)) = Ex_0$ by Corollary 2.6. Thus, $x_0 \in \mathcal{R}(\Sigma)$. $\Rightarrow$ There exist $\begin{bmatrix} x \\ u \end{bmatrix} \in C^{n+m}_{imp}$ such that $P_\Sigma(p)\begin{bmatrix} x \\ u \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $E(x(0^+)) = Ex_0$. If $x = x_1 + x_2$, $u = u_1 + u_2$, x_1 and u_1 impulsive, x_2 and u_2 smooth, then $pEx_2 = Ax_2 + Bu_2 + Ex_0$, $Cx_2 + Du_2 = 0$ (hence $x_0 \in \mathcal{V}(\Sigma)$) and $pE(-x_1) = A(-x_1) + B(-u_1) + Ex_0$, $C(-x_1) + D(-u_1) = 0$ (hence $x_0 \in \mathcal{W}(\Sigma)$). This completes the proof.

Remark 3.3.

Theorem 3.2 generalizes [23, Theorem 3.4] and [15, Propositions 3.23 and 3.25].

Of interest in the sequel is also the space $\mathcal{V}_C(\Sigma)$ of points x_0 for which there exist smooth x and u such that $P_\Sigma(p)\begin{bmatrix} x \\ u \end{bmatrix} = \begin{bmatrix} -Ex_0 \\ 0 \end{bmatrix}$ and $x(0^+) = x_0$. $\mathcal{V}_C(\Sigma)$ is a subspace of $\mathcal{V}(\Sigma)$. More precisely,

Proposition 3.4.

$$\mathcal{V}(\Sigma) = \mathcal{V}_C(\Sigma) + \ker(E).$$

Proof. $\Rightarrow$ Let $P_\Sigma(p)\begin{bmatrix}x\\u\end{bmatrix} = \begin{bmatrix}-Ex_0\\0\end{bmatrix}$, $\begin{bmatrix}x\\u\end{bmatrix}$ smooth. Then $x(0^+) \in \mathcal{V}_C(\Sigma)$ (yes, see (2.5b)!) and $x_0 - x(0^+) \in \ker(E)$ by Corollary 2.6. Thus, $x_0 = x(0^+) + (x_0 - x(0^+)) \in \mathcal{V}_C(\Sigma) + \ker(E)$. $\Leftarrow$ $\mathrm{Ker}(E) \subset \mathcal{R}(\Sigma)$.

We establish from Theorem 3.2 and Proposition 3.4 that $\mathcal{V}(\Sigma)$, $\mathcal{R}(\Sigma)$ and $\mathcal{V}_d(\Sigma)$ are known if $\mathcal{V}_C(\Sigma)$ and $\mathcal{W}(\Sigma)$ are. For these latter spaces we have the next statements and algorithms.

Proposition 3.5.

Let $\mathcal{L}$ be any subspace of $\mathbb{R}^n$. Then

$$\begin{bmatrix}A\\C\end{bmatrix}\mathcal{L} \subset \begin{bmatrix}E\mathcal{L}\\0\end{bmatrix} + \mathrm{im}(\begin{bmatrix}B\\D\end{bmatrix}) \Leftrightarrow$$

$$\exists_{F \in \mathbb{R}^{m\times n}}: (A + BF)\mathcal{L} \subset E\mathcal{L},\ (C + DF)\mathcal{L} = 0.$$

Proof. See e.g. the proof of [15, Theorem 3.10].

Theorem 3.6.

$\mathcal{V}_C(\Sigma)$ is the largest subspace $\mathcal{L}$ for which

$$\begin{bmatrix}A\\C\end{bmatrix}\mathcal{L} \subset \begin{bmatrix}E\mathcal{L}\\0\end{bmatrix} + \mathrm{im}(\begin{bmatrix}B\\D\end{bmatrix}).$$

Moreover, if $\mathcal{K}$ is any subspace of $\mathbb{R}^l$ such that

$$\mathcal{K} \subset E\begin{bmatrix}A\\C\end{bmatrix}^{-1}\{\begin{bmatrix}\mathcal{K}\\0\end{bmatrix} + \mathrm{im}(\begin{bmatrix}B\\D\end{bmatrix})\},\ \text{then } \mathcal{K} \subset E\mathcal{V}_C(\Sigma).$$

Proof. Without proof (compare e.g. [15, (3.12)]) we state

$$\forall_{x_0 \in \mathcal{V}_C(\Sigma)} \exists_{u_0 \in \mathbb{R}^m}: Ax_0 + Bu_0 \in E\mathcal{V}_C(\Sigma),\ Cx_0 + Du_0 = 0.$$

It follows that $\begin{bmatrix}A\\C\end{bmatrix}\mathcal{V}_C(\Sigma) \subset \begin{bmatrix}E\mathcal{V}_C(\Sigma)\\0\end{bmatrix} + \mathrm{im}(\begin{bmatrix}B\\D\end{bmatrix})$. Next, let $\mathcal{L}$ be any space such that for certain $F \in \mathbb{R}^{m\times n}$ (Proposition 3.5), $(A + BF)\mathcal{L} \subset E\mathcal{L}$, $(C + DF)\mathcal{L} = 0$. Then there exist a matrix K and a basis matrix L for $\mathcal{L}$ 1) such that $(A + BF)L = ELK$ and $(C + DF)L = 0$. Now, let $l = L\bar{x} \in \mathcal{L}$. By verification we establish that $P_\Sigma(p)\begin{bmatrix}x\\u\end{bmatrix} = \begin{bmatrix}-El\\0\end{bmatrix}$ with $\begin{bmatrix}x\\u\end{bmatrix} = \begin{bmatrix}I\\F\end{bmatrix}L(pI - K)^{-1}\bar{x}$ and $x(0^+) = L\bar{x} = l$ [15, p. 375]. Hence $l \in \mathcal{V}_C(\Sigma)$. This proves the first claim.

Next, we have $\mathcal{X} = E(E^{-1}\mathcal{X})$ since $\mathcal{X} \subset \mathrm{im}(E)$ (always $E(E^{-1}\mathcal{X}) \subset \mathcal{X}$). Now, assume that $E\mathcal{N} \subset E\mathcal{M}$ with $\mathcal{M} := \begin{bmatrix} A \\ C \end{bmatrix}^{-1} \{ \begin{bmatrix} E\mathcal{N} \\ 0 \end{bmatrix} + \mathrm{im}(\begin{bmatrix} B \\ D \end{bmatrix}) \}$. Then $\mathcal{N} \subset \mathcal{M} + \ker(E)$. In addition, $\mathcal{M} \subset \begin{bmatrix} A \\ C \end{bmatrix}^{-1} \{ \begin{bmatrix} E\mathcal{M} \\ 0 \end{bmatrix} + \mathrm{im}(\begin{bmatrix} B \\ D \end{bmatrix}) \}$, i.e., $\begin{bmatrix} A \\ C \end{bmatrix}\mathcal{M} \subset \begin{bmatrix} E\mathcal{M} \\ 0 \end{bmatrix} + \mathrm{im}(\begin{bmatrix} B \\ D \end{bmatrix})$, since $E\mathcal{N} \subset E\mathcal{M}$. But then, by the foregoing, $\mathcal{M} \subset \mathcal{V}_C(\Sigma)$ and hence $\mathcal{N} \subset \mathcal{V}_C(\Sigma) + \ker(E)$ and $E\mathcal{N} \subset E\mathcal{V}_C(\Sigma)$. Taking $\mathcal{N} = E^{-1}\mathcal{X}$ completes the proof ([1] this observation was found in [25]).

Remark 3.7.

Our space $\mathcal{V}_C(\Sigma)$ corresponds to the so-called supremal Output Nulling (A, E, im(B))-invariant subspace of [25] - however, we do not require sE - A to be invertible. If D = 0, then $\mathcal{V}_C(\Sigma)$ equals the supremal (A, E, B)-invariant subspace $\mathcal{V}^*$ in [10, Sec. 2] (see Proposition 3.8) - yet, unlike as in [10, Sec. 3], we allow sE - A to be arbitrary in our dynamical subspace interpretations.

Proposition 3.8 contains the same Molinari-type algorithm [26] for the construction of $\mathcal{V}_C(\Sigma)$ as e.g. [10].

Proposition 3.8.

Consider the algorithm

$$\mathcal{V}_0 := \mathbb{R}^n, \ \mathcal{V}_{i+1} := \begin{bmatrix} A \\ C \end{bmatrix}^{-1} \{ \begin{bmatrix} E\mathcal{V}_i \\ 0 \end{bmatrix} + \mathrm{im}(\begin{bmatrix} B \\ D \end{bmatrix}) \}.$$

Then $\mathcal{V}_0 \supset \mathcal{V}_1 \supset \ldots \supset \mathcal{V}_i \supset \mathcal{V}_{i+1} \supset \ldots \supset \mathcal{V}_n = \mathcal{V}_C(\Sigma)$.

Proof. The inclusion is clear by induction. Next, we have $\mathcal{V}_C(\Sigma) \subset \mathcal{V}_i$ for all i, since if $\mathcal{V}_i \supset \mathcal{V}_C(\Sigma)$, then $\mathcal{V}_{i+1} \supset \mathcal{V}_C(\Sigma)$ by Theorem 3.6. Now assume that $\mathcal{V}_i = \mathcal{V}_{i+1}$. Then $\mathcal{V}_i \subset \mathcal{V}_C(\Sigma)$, again by Theorem 3.6. It follows that $\mathcal{V}_i = \mathcal{V}_C(\Sigma)$ and thus $\mathcal{V}_n = \mathcal{V}_C(\Sigma)$.

Next, we investigate $\mathcal{W}(\Sigma)$.

Theorem 3.9.

$\mathcal{W}(\Sigma)$ is the smallest subspace $\mathcal{L}$ for which

$$E^{-1}[A\ B]\{(\mathcal{L} \oplus \mathbb{R}^m) \cap \ker([C\ D])\} \subset \mathcal{L}. \qquad (3.2)$$

Proof. Assume that x_0 is such that $Ex_0 = Aw + Bu_0$ with $Cw + Du_0 = 0$, $u_0 \in \mathbb{R}^m$ and $w \in \mathcal{W}(\Sigma)$. There exist impulsive u_1 and x_1 such that $pEx_1 = Ax_1 + Bu_1 + Ew$, $Cx_1 + Du_1 = 0$, by definition of $\mathcal{W}(\Sigma)$. Now, define $\bar{u} := pu_1 - u_0$, impulsive, and $\bar{x} := px_1 - w$, impulsive. Then $pE\bar{x} = A\bar{x} + B\bar{u} + Ex_0$, $C\bar{x} + D\bar{u} = 0$, i.e., $x_0 \in \mathcal{W}(\Sigma)$. Next, let $\mathcal{L} \subset \mathbb{R}^n$ satisfy (3.2) and let $x_0 \in \mathcal{W}(\Sigma)$. Then there exist impulsive u_1 and x_1 such that $pEx_1 = Ax_1 + Bu_1 + Ex_0$ and $Cx_1 + Du_1 = 0$. Suppose $u_1 = \sum_{i=0}^{k} \beta_i p^i$ and $x_1 = \sum_{i=0}^{k+j} \alpha_i p^i$ with α_i, β_i real column vectors and $j \geq 0$. Then $E\alpha_{k+j} = 0$, $E\alpha_{k+j-1} = A\alpha_{k+j}$, $C\alpha_{k+j} = 0$, ..., $E\alpha_k = A\alpha_{k+1}$, $C\alpha_{k+1} = 0$, $E\alpha_{k-1} = A\alpha_k + B\beta_k$, $C\alpha_k + D\beta_k = 0$, ..., $E\alpha_0 = A\alpha_1 + B\beta_1$, $C\alpha_1 + D\beta_1 = 0$, $0 = A\alpha_0 + B\beta_0 + Ex_0$, $C\alpha_0 + D\beta_0 = 0$. Hence $\alpha_{k+j} \in \mathcal{L}$, $\alpha_{k+j-1} \in \mathcal{L}$, ..., $\alpha_k \in \mathcal{L}$, $\alpha_{k-1} \in \mathcal{L}$, ..., $\alpha_0 \in \mathcal{L}$ and $x_0 \in \mathcal{L}$. If $j = -k, \ldots, -1$, the proof runs similarly.

Theorem 3.10.

Consider the algorithm

$$\mathcal{W}_0 := \ker(E),$$

$$\mathcal{W}_{i+1} := E^{-1}[A\ B]\{(\mathcal{W}_i \oplus \mathbb{R}^m) \cap \ker([C\ D])\}.$$

Then $\mathcal{W}_0 \subset \mathcal{W}_1 \subset \ldots \subset \mathcal{W}_n = \mathcal{W}(\Sigma)$.

Proof. Since $\ker(E) \subset \mathcal{W}_1$, the inclusions are clear by induction. Also, $\mathcal{W}_0 \subset \mathcal{W}(\Sigma)$. Now, suppose that $\mathcal{W}_i \subset \mathcal{W}(\Sigma)$. Then $\mathcal{W}_{i+1} \subset \mathcal{W}(\Sigma)$ by Theorem 3.9 and thus all $\mathcal{W}_i \subset \mathcal{W}(\Sigma)$. If $\mathcal{W}_i = \mathcal{W}_{i+1}$, then $\mathcal{W}_i \subset \mathcal{W}(\Sigma) \subset \mathcal{W}_i$ (Theorem 3.9) and thus $\mathcal{W}_i = \mathcal{W}(\Sigma)$ - in particular, $\mathcal{W}_n = \mathcal{W}(\Sigma)$ since our system is finite-dimensional.

Remark 3.11.

Our subspace $\mathcal{W}(\Sigma)$ is the generalization of Malabre's $\mathcal{S}^*$ in [10, Definition 12], where sE - A is assumed invertible. If D = 0, $\mathcal{W}(\Sigma)$ may be called the infimal (C, A, E)-invariant subspace related to im(B) [10], see also Corollary 3.13. Note that every point in $\mathcal{W}_i$ (Theorem 3.10) can be "controlled impulsively" by an impulsive $\begin{bmatrix} u \\ x \end{bmatrix} = \mathcal{P}(p)$, where $\mathcal{P}(s)$ is polynomial of degree $\leq$ i-1 (and a polynomial of degree -1 is assumed to be zero). This follows directly from the proof of Theorem 3.9. In terms of Willems [27], $\mathcal{W}(\Sigma)$ stands for the controllable L_2-almost output nulling subspace and $\mathcal{V}_d(\Sigma)$ stands for the L_2-almost output nulling subspace. Our $\mathcal{R}(\Sigma)$ corresponds to Willems' controllable output nulling subspace. See also [6, p. 1291].

There exist certain duality (see e.g. [22, Ch. 0.12]) results between $\mathcal{V}_C(\Sigma)$ and $\mathcal{W}(\Sigma)$, but not the usual ones [15, p. 380] of course, as l may be unequal to n. Theorem 3.12 generalizes duality statements in [10], since we start from open-loop subspace definitions (Definition 3.1) rather than from algebraic representations as Theorems 3.6 and 3.9.

Theorem 3.12.

Let $\Sigma' := (E', A', C', B', D')$. Then

$$\mathcal{W}(\Sigma) = (E'\mathcal{V}_C(\Sigma'))^{\perp} = E^{-1}(\mathcal{V}_C(\Sigma'))^{\perp}$$

and $(\mathcal{W}(\Sigma'))^{\perp} = E\mathcal{V}_C(\Sigma)$.

Proof. According to Theorem 3.6, we have

$$E'\mathcal{V}_C(\Sigma') \subset E'\begin{bmatrix} A' \\ B' \end{bmatrix}^{-1}\left\{\begin{bmatrix} E'\mathcal{V}_C(\Sigma') \\ 0 \end{bmatrix} + \mathrm{im}\left(\begin{bmatrix} C' \\ D' \end{bmatrix}\right)\right\},$$

and hence $\mathcal{K} := (E'\mathcal{V}_C(\Sigma'))^{\perp} \supset E^{-1}[A\ B]\{(\mathcal{K} \oplus \mathbb{R}^m) \cap \ker([C\ D])\}$.

Thus, $\mathscr{W}(\Sigma) \subset \mathscr{X}$ by Theorem 3.9, i.e., $E'\mathscr{V}_C(\Sigma') \subset (\mathscr{W}(\Sigma))^{\perp}$. On the other hand, $(\mathscr{W}(\Sigma))^{\perp} \subset E'\begin{bmatrix} A' \\ B' \end{bmatrix}^{-1}\{\begin{bmatrix} (\mathscr{W}(\Sigma))^{\perp} \\ 0 \end{bmatrix} + \mathrm{im}(\begin{bmatrix} C' \\ D' \end{bmatrix})\}$, again by Theorem 3.9, and hence $(\mathscr{W}(\Sigma))^{\perp} \subset E'\mathscr{V}_C(\Sigma')$ by the last claim of Theorem 3.6! It follows that $(\mathscr{W}(\Sigma))^{\perp} = E'\mathscr{V}_C(\Sigma')$ and thus $\mathscr{W}(\Sigma) = (E'\mathscr{V}_C(\Sigma))^{\perp} = E^{-1}(\mathscr{V}_C(\Sigma'))^{\perp}$. Hence also $\mathscr{W}(\Sigma') = (E\mathscr{V}_C(\Sigma))^{\perp}$.

Corollary 3.13.

$\mathscr{W}(\Sigma)$ is the smallest subspace $\mathscr{L}$ for which there exists a matrix $G \in \mathbb{R}^{l\times r}$ such that

$$E^{-1}\{(A + GC)\mathscr{L} + \mathrm{im}(B + GD)\} \subset \mathscr{L}.$$

Proof. By Proposition 3.5 and Theorem 3.6, there exists a $G' \in \mathbb{R}^{r\times l}$ such that

$$(A' + C'G')\mathscr{V}_C(\Sigma') \subset E'\mathscr{V}_C(\Sigma'),\quad (B' + D'G')\mathscr{V}_C(\Sigma') = 0.$$

Hence, by Theorem 3.12, $(A + GC)\mathscr{W}(\Sigma) \subset (\mathscr{V}_C(\Sigma'))^{\perp}$, $\mathrm{im}(B + GD) \subset (\mathscr{V}_C(\Sigma'))^{\perp}$, i.e., $(A + GC)\mathscr{W}(\Sigma) + \mathrm{im}(B + GD) \subset (\mathscr{V}_C(\Sigma'))^{\perp}$ and thus $E^{-1}\{(A + GC)\mathscr{W}(\Sigma) + \mathrm{im}(B + GD)\} \subset \mathscr{W}(\Sigma)$; $\mathscr{W}(\Sigma)$ satisfies the claim. Next, let $\mathscr{L} \subset \mathbb{R}^n$ and $G \in \mathbb{R}^{l\times r}$ such that

$$\mathscr{L}^{\perp} \subset E'\{(A' + C'G')^{-1}\mathscr{L}^{\perp} \cap \ker(B' + D'G')\},$$

then $\mathscr{L}^{\perp} \subset E'\begin{bmatrix} A' \\ B' \end{bmatrix}^{-1}\{\begin{bmatrix} \mathscr{L}^{\perp} \\ 0 \end{bmatrix} + \mathrm{im}(\begin{bmatrix} C' \\ D' \end{bmatrix})\}$ and hence $\mathscr{L}^{\perp} \subset E'\mathscr{V}_C(\Sigma')$ (last statement of Theorem 3.6), i.e., $\mathscr{L} \supset \mathscr{W}(\Sigma)$ (Theorem 3.12).

In this Section we have defined 5 different subspaces in terms of distributions and we have seen how they can be computed - note, that all results reduce directly to corresponding ones in [15] if $E = I$.

In the final Section 4 we will define our concepts of singular system invertibility and relate these notions to the subspaces as well as to the system matrix.

4. System invertibility.

Invertibility concepts in terms of distributions for *standard* systems, i.e., systems with $E = I$, were introduced in [15, Section 3], see also [27]. Now we propose the following straightforward generalizations of these concepts for an arbitrary singular system Σ of the form

$$pEx = Ax + Bu + Ex_0, \quad y = Cx + Du, \tag{4.1}$$

with $(x_0, u) \in \mathbb{R}^n \times C^m_{imp}$. As in Corollary 2.4, we denote the set of $k_1 \times k_2$ matrices with elements in $\mathbb{R}(s)$, the field of rational functions, by $M^{k_1 \times k_2}(s)$.

Definition 4.1.

A system $\Sigma = (E, A, B, C, D)$ is called left invertible in the *weak* sense if

$$x_0 = 0 \text{ and } y = 0 \Rightarrow u = 0.$$

Theorem 4.2.

Σ is left invertible in the weak sense if and only if for every $\begin{bmatrix} x(s) \\ u(s) \end{bmatrix} \in M^{(n+m)\times 1}(s)$,

$$P_\Sigma(s) \begin{bmatrix} x(s) \\ u(s) \end{bmatrix} = 0 \Leftrightarrow \begin{bmatrix} A - sE \\ C \end{bmatrix} x(s) = 0, \ u(s) = 0.$$

Proof. $\Rightarrow$ Assume that (Proposition 2.3) $P_\Sigma(p) \begin{bmatrix} x(p) \\ u(p) \end{bmatrix} = 0$. Then, by definition, $u(p) = 0$ and also $\begin{bmatrix} A - pE \\ C \end{bmatrix} x(p) = 0$. $\Leftarrow$ Assume without loss of generality that $\begin{bmatrix} A - sE \\ C \end{bmatrix} = \begin{bmatrix} Q_1(s) \\ Q_2(s) \end{bmatrix} [I_{n_1} \ \ X(s)]$ with $\begin{bmatrix} Q_1(s) \\ Q_2(s) \end{bmatrix} \in M^{(1+r)\times n_1}(s)$, left invertible as a rational matrix, $X(s) \in M^{n_1 \times (n-n_1)}(s)$. Then the claim is equivalent to left-invertibility of $\begin{bmatrix} Q_1(s) & B \\ Q_2(s) & D \end{bmatrix}$ (Proof: Let $\begin{bmatrix} Q_1(s) & B \\ Q_2(s) & D \end{bmatrix} \begin{bmatrix} x_1(s) \\ u(s) \end{bmatrix} = 0$, then $\begin{bmatrix} Q_1(s) \\ Q_2(s) \end{bmatrix} [I_{n_1} \ \ X(s)] \begin{bmatrix} x_1(s) \\ 0 \end{bmatrix} + \begin{bmatrix} B \\ D \end{bmatrix} u(s) = 0$ and hence $u(s) =$

0, $x_1(s) = 0$. Conversely, let $\begin{bmatrix} Q_1(s) \\ Q_2(s) \end{bmatrix}[I_{n_1} \;\; X(s)]x(s) + \begin{bmatrix} B \\ D \end{bmatrix}u(s) = 0$ then $[I_{n_1} \;\; X(s)]x(s) = 0$, $u(s) = 0$, and thus $\begin{bmatrix} A - sE \\ C \end{bmatrix}x(s) = 0$, $u(s) = 0$.) Hence, if $P_\Sigma(p)\begin{bmatrix} x \\ u \end{bmatrix} = 0$ for certain $\begin{bmatrix} x \\ u \end{bmatrix} \in C_{imp}^{n+m}$, then $\begin{bmatrix} Q_1(p) \\ Q_2(p) \end{bmatrix}[I_{n_1} \;\; X(p)]x + \begin{bmatrix} B \\ D \end{bmatrix}u = 0$. Let $L(s) \in M^{(n_1+m)\times(l+r)}(s)$ be a left inverse of $\begin{bmatrix} Q_1(s) & B \\ Q_2(s) & D \end{bmatrix}$, then (Corollary 2.4) $[I_{n_1} \;\; X(p)]x = 0$, $u = 0$ since C_{imp} is a commutative ring. This completes the proof.

Definition 4.3.

A system $\Sigma = (E, A, B, C, D)$ is called right invertible in the *weak* sense if

$$\forall_{\bar{y} \in C_{imp}^r} \exists_{u \in C_{imp}^m} \exists_{x \in S(0,\, u)} : y = \bar{y}.$$

Theorem 4.4.

Σ is right invertible in the weak sense if and only if for every $[\eta(s) \;\; \xi(s)] \in M^{1\times(l+r)}(s)$,

$$[\eta(s) \;\; \xi(s)]P_\Sigma(s) = 0 \Leftrightarrow \eta(s)[A - sE \;\; B] = 0,\ \xi(s) = 0.$$

Proof. $\Rightarrow$ Assume that (Proposition 2.3) $[\eta(p) \;\; \xi(p)]P_\Sigma(p) = 0$. Since for every standard basis vector e_i in $\mathbb{R}^l$ ($i = 1, \dots, l$) there exists a $\begin{bmatrix} x_i \\ u_i \end{bmatrix} \in C_{imp}^{n+m}$ such that $P_\Sigma(p)\begin{bmatrix} x_i \\ u_i \end{bmatrix} = \begin{bmatrix} 0 \\ e_i \end{bmatrix}$ (with e_i now standing for $e_i\delta$!), we find that $\xi(p) = 0$ and thus also $\eta(p)[A - pE \;\; B] = 0$. $\Leftarrow$ Dualize the second part of Theorem 4.2.

Remark 4.5.

Fully independently, several kinds of invertibility were defined and characterized for *discrete-time* singular systems in [29]. Apparently, left (right) invertibility in [29] coincides with our left (right) invertibility in the weak sense (compare [29, Corollaries 3.1, 4.1] with our Theorems 4.2, 4.4), although our definitions for *continuous-time* systems are given in terms of distributions. However, one should recall in this context that left (right) invertibility for standard systems (= left (right) invertibility of the associated transfer function) was formulated within a distributional framework earlier [15]. Finally, observe that weak left and weak right invertibility are *dual* concepts.

Weak right invertibility can also be quantified with the set $\mathcal{T}$ of points x_0 from where every $\bar{y} \in C^r_{imp}$ is attainable:

$$\mathcal{T} := \{x_0 \in \mathbb{R}^n \mid \forall_{\bar{y} \in C^r_{imp}} \exists_{u \in C^m_{imp}} \exists_{x \in S(x_0,\, u)} : y = \bar{y}\}. \tag{4.2}$$

It is clear that $\mathcal{T} \subset \mathcal{V}_d(\Sigma)$, the distributionally weakly unobservable subspace. The converse is true if and only if Σ is right invertible in the weak sense, i.e., if $0 \in \mathcal{T}$.

Theorem 4.6.

Σ is right invertible in the weak sense if and only if $\mathcal{T} = \mathcal{V}_d(\Sigma)$.

Proof. $\Rightarrow$ Let $x_0 \in \mathcal{V}_d(\Sigma)$, i.e., let $\begin{bmatrix} x_1 \\ u_1 \end{bmatrix} \in C^{n+m}_{imp}$ be such that $P_\Sigma(p)\begin{bmatrix} x_1 \\ u_1 \end{bmatrix} = \begin{bmatrix} -Ex_0 \\ 0 \end{bmatrix}$ and let $\bar{y} \in C^r_{imp}$. Then there also exists a $\begin{bmatrix} x_2 \\ u_2 \end{bmatrix}$ such that $P_\Sigma(p)\begin{bmatrix} x_2 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ \bar{y} \end{bmatrix}$. It follows that $P_\Sigma(p)\begin{bmatrix} x \\ u \end{bmatrix} = \begin{bmatrix} -Ex_0 \\ \bar{y} \end{bmatrix}$ with $\begin{bmatrix} x \\ u \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ u_1 + u_2 \end{bmatrix}$ and hence $x_0 \in \mathcal{T}$. $\Leftarrow$ $0 \in \mathcal{V}_d(\Sigma)$.

The case $\mathcal{T} = \mathbb{R}^n$ turns out to be of special interest.

Definition 4.7.

A system Σ is called right invertible in the *strong* sense if

$$\forall_{x_0 \in \mathbb{R}^n} \forall_{\bar{y} \in C^r_{imp}} \exists_{u \in C^m_{imp}} \exists_{x \in S(x_0, u)} : y = \bar{y}.$$

Proposition 4.8.

Σ is right invertible in the strong sense if and only if Σ is right invertible in the weak sense and $\mathcal{V}_d(\Sigma) = \mathbb{R}^n$.

Proof. $\Rightarrow$ $\mathcal{T} = \mathbb{R}^n \subset \mathcal{V}_d(\Sigma) \subset \mathbb{R}^n$. $\Leftarrow$ From Theorem 4.6, $\mathcal{T} = \mathbb{R}^n$.

If sE - A is invertible, then, according to Proposition 4.8 and [23, Theorem 3.8], weak right invertibility implies strong right invertibility (see also [15, Theorem 3.24] for the case E = I). In general, however, this is not the case. More precisely,

Theorem 4.9.

The following statements are equivalent.

i) Σ is right invertible in the strong sense.

ii) $\mathcal{V}_d(\Sigma) = \mathbb{R}^n$,

$$\forall_{[\eta(s)\ \ \xi(s)] \in M^{1\times(l+r)}(s)}:$$
$$[\eta(s)\ \ \xi(s)]\begin{bmatrix} E & A & B \\ 0 & C & D \end{bmatrix} = 0 \Leftrightarrow \eta(s)[E\ A\ B] = 0,\ \xi(s) = 0.$$

iii) $\forall_{[\eta(s)\ \ \xi(s)] \in M^{1\times(l+r)}(s)}$:
$$[\eta(s)\ \ \xi(s)]P_\Sigma(s) = 0 \Leftrightarrow \eta(s)[E\ A\ B] = 0,\ \xi(s) = 0.$$

Proof. i) $\Rightarrow$ ii). For every standard basis vector e_i in $\mathbb{R}^l$ there exist u_i and x_i in C^m_{imp} and C^n_{imp}, respectively, such that $pEx_i = Ax_i + Bu_i$, $e_i = Cx_i + Du_i$. If $[\eta(s)\ \ \xi(s)]\begin{bmatrix} E & A & B \\ 0 & C & D \end{bmatrix} = 0$, then 0 =

$\eta(p)pEx_i = \eta(p)[Ax_i + Bu_i] = -\xi(p)[Cx_i + Du_i] = -\xi(p)e_i$ (Proposition 2.3) and hence $\xi(p) = 0$. ii) $\Rightarrow$ iii). Assume that $[\eta(s)\ \ \xi(s)]P_\Sigma(s) = 0$. Since $\mathcal{V}_d(\Sigma) = \mathbb{R}^n$, it follows that $\eta(p)Ex_0 = 0$ for all x_0, i.e., $\eta(p)E = 0$. Thus, by ii) and Proposition 2.3,, $\eta(s)[E\ A\ B] = 0$ and $\xi(s) = 0$. iii) $\Rightarrow$ i). Without loss of generality, assume that $[E\ A\ B] = \begin{bmatrix} I_{l_1} \\ Y \end{bmatrix}[T_1\ T_2\ T_3]$ with $T_i \in \mathbb{R}^{l_1 \times n}$ $(i = 1, 2)$, $T_3 \in \mathbb{R}^{l_1 \times m}$, $Y \in \mathbb{R}^{(l-l_1)\times l_1}$, $[T_1\ T_2\ T_3]$ right invertible. Then it follows that $\begin{bmatrix} T_2 - sT_1 & T_3 \\ C & D \end{bmatrix}$ is right invertible (compare second part of proof of Theorem 4.2). If $R(s)$ is any right inverse, then for every $x_0 \in \mathbb{R}^n$, $\bar{y} \in C^r_{imp}$ it can be easily seen that $P_\Sigma(p)\begin{bmatrix} x \\ u \end{bmatrix} = \begin{bmatrix} -Ex_0 \\ \bar{y} \end{bmatrix}$ with $\begin{bmatrix} x \\ u \end{bmatrix} := R(p)\begin{bmatrix} -T_1x_0 \\ \bar{y} \end{bmatrix}$. This completes the proof.

Not surprisingly, the dual counterpart of strong **right** invertibility will be called strong **left** invertibility.

Definition 4.10.

A system Σ will be called left invertible in the *strong* sense if

$$x_0 = 0,\ y = 0 \Rightarrow u = 0,\ Ex = 0.$$

Theorem 4.11.

The following statements are equivalent.

i) Σ is left invertible in the strong sense.

ii) $\mathcal{R}(\Sigma) = \ker(E)$,

$$\forall \begin{bmatrix} x(s) \\ u(s) \end{bmatrix} \in M^{(n+m)\times 1}(s):$$
$$\begin{bmatrix} E & 0 \\ A & B \\ C & D \end{bmatrix}\begin{bmatrix} x(s) \\ u(s) \end{bmatrix} = 0 \Leftrightarrow \begin{bmatrix} E \\ A \\ C \end{bmatrix} x(s) = 0,\ u(s) = 0.$$

iii) $\forall \begin{bmatrix} x(s) \\ u(s) \end{bmatrix} \in M^{(n+m)\times 1}(s):$

$$P_\Sigma(s)\begin{bmatrix} x(s) \\ u(s) \end{bmatrix} = 0 \Leftrightarrow \begin{bmatrix} E \\ A \\ C \end{bmatrix} x(s) = 0,\ u(s) = 0.$$

Proof. i) $\Rightarrow$ iii). By Proposition 2.3, we establish that $u(p) = 0$, $\begin{bmatrix} A - pE \\ C \end{bmatrix} x(p) = 0$ and $Ex(p) = 0$. iii) $\Rightarrow$ i). We may write $\begin{bmatrix} E \\ A \\ C \end{bmatrix} = \begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \end{bmatrix} [I \ X]$ with $\begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \end{bmatrix}$ left invertible. As earlier, we establish that $\begin{bmatrix} Q_2 - sQ_1 & B \\ Q_3 & D \end{bmatrix}$ is left invertible. Now let $P_\Sigma(p) \begin{bmatrix} x \\ u \end{bmatrix} = 0$, $\begin{bmatrix} x \\ u \end{bmatrix} \in C^{n+m}_{imp}$, i.e., let $\begin{bmatrix} Q_2 - sQ_1 \\ Q_3 \end{bmatrix} [I \ X]x + \begin{bmatrix} B \\ D \end{bmatrix} u = 0$. Then $[I \ X]x = 0$, $u = 0$ and hence $Ex = 0$, $u = 0$. ii) $\Leftrightarrow$ iii). If $\Sigma' := (E', A', C', B', D')$, then it follows from the above and Theorem 4.9 that Σ' is strongly right invertible if and only if Σ is strongly left invertible. Since $\mathcal{V}(\Sigma') + \mathcal{W}(\Sigma') = \mathbb{R}^l \Leftrightarrow \mathcal{V}_C(\Sigma') + \mathcal{W}(\Sigma') = \mathbb{R}^l$ (Proposition 3.4) $\Leftrightarrow (\mathcal{V}_C(\Sigma'))^\perp \cap E\mathcal{V}_C(\Sigma) = 0$ (Theorem 3.12) $\Leftrightarrow (\mathcal{V}_C(\Sigma'))^\perp \cap E\mathcal{V}(\Sigma) = 0 \Leftrightarrow E^{-1}(\mathcal{V}_C(\Sigma'))^\perp \cap \mathcal{V}(\Sigma) = \ker(E) \Leftrightarrow \mathcal{W}(\Sigma) \cap \mathcal{V}(\Sigma) = \mathcal{X}(\Sigma) = \ker(E)$ (Theorems 3.12, 3.2), the proof is now complete.

Proposition 4.12.

Σ is left invertible in the strong sense if and only if Σ is left invertible in the weak sense and $\mathcal{X}(\Sigma) = \ker(E)$.

Proof. $\Sigma' := (E', A', C', B', D')$ is strongly right invertible if and only if Σ' is weakly right invertible and $\mathcal{V}_d(\Sigma') = \mathbb{R}^l$, by Proposition 4.8.

Hence weak and strong left invertibility are equivalent, just as weak and strong right invertibility are, if $sE - A$ is invertible [23, Theorem 3.9]. Our final Corollaries consider two more general situations where weak and strong left (right) invertibility are equivalent.

Corollary 4.13.

Assume that [E A B] is of full row rank. Then the following statements are equivalent.

i) Σ is right invertible in the strong sense.

ii) $\mathcal{V}_d(\Sigma) = \mathbb{R}^n$, $\begin{bmatrix} E & A & B \\ 0 & C & D \end{bmatrix}$ is of full row rank.

iii) $P_\Sigma(s)$ is right invertible as a rational matrix.

Moreover,

$$[A - sE \quad B] \text{ right invertible} \Leftrightarrow$$

$$\forall_{x_0 \in \mathbb{R}^n} \exists_{u \in C^m_{imp}} : S(x_0, u) \neq \emptyset,$$

amd if $[A - sE \quad B]$ is right invertible, then weak and strong right invertibility are equivalent.

Proof. The first claim is immediate from Theorem 4.9. If $R(s) = \begin{bmatrix} R_1(s) \\ R_2(s) \end{bmatrix}$ is a right inverse of $[A - sE \quad B]$, then $u := R_2(p)(-Ex_0)$ is such that $x := R_1(p)(-Ex_0) \in S(x_0, u)$. Conversely, assume that $\eta(s)[A - sE \quad B] = 0$ ($\eta(s)$ rational), then $\eta(p)Ex_0 = 0$ for all x_0 and hence $\eta(p) = 0$. Finally, apply Theorem 4.4.

Remark 4.14.

In [13, Definition 2.4], [30, Definition 3.1] the system $pEx = Ax + Bu + Ex_0$ is called (C)-ontrol solvable if

$$\forall_{x_0 \in \mathbb{R}^n} \exists_{u \in C^m_{imp}} : S(x_0, u) \neq \emptyset.$$

Indeed, if for a certain x_0, $S(x_0, u) = \emptyset$ for all u, then e.g. linear-quadratic optimal control problems [15], [27], [21], [8], [28], [31] are not well posed. Since one may assume without loss of generality that [E A B] is of full row rank in (2.4a), we observe from Corollary 4.13 that right invertibility (in either sense) is equivalent to right invertibility of Rosenbrock's system matrix if the system (2.4a) is C-solvable.

Corollary 4.15.

Assume that $\begin{bmatrix} E \\ A \\ C \end{bmatrix}$ is of full column rank. Then the following statements are equivalent.

i) Σ is left invertible in the strong sense.

ii) if $x_0 = 0$ and $y = 0$, then $u = 0$, $x = 0$.

iii) $\mathfrak{X}(\Sigma) = \ker(E)$, $\begin{bmatrix} E & 0 \\ A & B \\ C & D \end{bmatrix}$ is of full column rank.

iv) $P_\Sigma(s)$ is left invertible as a rational matrix.

Moreover, if $\begin{bmatrix} A - sE \\ C \end{bmatrix}$ is left invertible, then weak and strong left invertibility are equivalent.

Proof. Straightforward, by dualizing Corollary 4.13; observe that, if $P_\Sigma(p)\begin{bmatrix} x \\ u \end{bmatrix} = 0$ yields $u = 0$ and $Ex = 0$, then also $\begin{bmatrix} E \\ A \\ C \end{bmatrix}x = 0$ and hence $u = 0$, $x = 0$.

Remark 4.16.

Observe that, if $sE - A$ is invertible, then left (right) invertibility of the system matrix $P_\Sigma(s)$ is equivalent to left (right) invertibility of $T(s) = D + C(sE - A)^{-1}B$, the transfer function of Σ [23, Theorems 3.8, 3.9].

Remark 4.17.

In the recent [16], conditions for left and right invertibility of singular systems were given in the case of existence of the transfer function, that naturally arises when starting from a realizational point of view, that is, when one tries to find a suitable state-space representation for a linear system given by autoregressive equations.

Here, however, we consider the "reversed" situation: A system is given in state-space form as a result of its mere nature (an electrical circuit or an econometrical model, for instance) and one is interested in the system's behaviour under the influence of diverse control inputs. Moreover, we do not require the transfer function to exist. For example, if $\Sigma = (0, 0, I, I, 0)$, then the transfer function does not exist according to [16, Theorem 4.3], whereas in our context the (pathological) system Σ is both left and right invertible in the strong sense.

5. Conclusions.

By means of our fully algebraic distributional framework and without any assumptions on the coefficients of the singular system $\Sigma = (E, A, B, C, D)$, we have defined and characterized in full detail

several subspaces of interest (e.g. with respect to optimal control problems) and their relative connections, and

several concepts of left and right invertibility for the system Σ and the 'gaps' between these notions.

Moreover, we have proven various relations between these subspaces, the concepts of invertibility and Rosenbrock's system matrix.

In future papers such as [31] we hope to present a complete treatment of general linear-quadratic optimal control problems subject to general linear systems along the lines of the distributional approach and the results exposed here and in [30].

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